# A meshfree isovalue search method for boundary element methods

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**An efficient isovalue search method is a prerequisite for the computation of isosurfaces, which are an established approach to visualize scalar fields or the absolute value of vector fields in three dimensions. A classical method to make volume data available in the case of boundary element methods is to precompute the values of the examined field in the nodes of an auxiliary post-processing volume mesh and to interpolate the field values with linear shape functions. Then, an octree scheme is applied to find volume elements, which are intersected by the isosurface. Finally, the surface elements of the isosurface are constructed using the intersection points of the isosurface with the volume elements. The accuracy and the computational costs are mainly influenced by the density of the volume mesh. Here, the applicability of a meshfree approach for the isovalue search is investigated. An octree-based search method is directly coupled to the octree-based fast multipole method. Then, needed field values are obtained from an evaluation of higher order polynomials in spherical coordinates. Thus, a high accuracy of the isovalue search is achieved along with relatively low computational costs.**

***Index Terms*—boundary element methods, fast multipole methods, isosurfaces, meshfree methods**

## I. Introduction

The computation of isosurfaces is a very attractive method to detect and visualize domains with a given range of scalar field values in three dimensions. Furthermore, the characteristic properties of an examined field are sufficiently described by only a few isosurfaces in many practical applications. However, an efficient isovalue search is still a challenging task. On the one hand, field values must be available in total space. A classical approach to provide volume data in the case of boundary element methods (BEM) is to compute the examined field values in the nodes of an auxiliary post-processing mesh. On the other hand, fast search methods are necessary to find isovalues efficiently. An established isovalue search method is an octree-based scheme [1]. A pointer-less organization of the octree enables very efficient storage of the tree along with fast navigation in the tree in the case of dense octrees [1]. Further enhancements improve the accuracy in the case of complex shaped isosurfaces in combination with unstructured grids for field values [2, 3, 4]. It is also possible to compute not only correct isosurfaces but to obtain surface elements, which can be used for further field computations [5]. Modern implementations of isovalue search methods run massively parallel on graphics processing units (GPU) [6]. However, the total accuracy and efficiency of all these approaches depends significantly on the density of the post-processing volume mesh for the field values. There, a compromise between accuracy and computational costs must be found to obtain correct and smooth isosurfaces.

The computation of field values in a large number of evaluation points is an expensive task if classical BEM is used. Fortunately, compression techniques like the fast multipole method (FMM) [7] reduce the computational costs significantly both for the solution of the linear system of equations and for the post-processing [8]. Furthermore, it is possible to compute very efficiently field values for visualization objects, for instance streamlines, without an auxiliary mesh by a bidirectional coupling of BEM with visualization approaches [9].

Here, an innovative coupling of a meshfree BEM post-processing with an efficient isovalue search is presented. The basis is an adaptive octree scheme, which is used both for the isovalue search and for a FMM accelerated BEM computation of field values. A direct evaluation of the series expansions of the FMM enables very accurate field value computations in relatively large domains using only a small number of coefficients. Hence, intersection points of isosurfaces with octree cubes can be determined more precisely and expensive computations of field values in the nodes of a dense mesh are avoided.

## II. Numerical Formulation

### A. Fast Boundary Element Method

Here, three-dimensional problems, which are based on a solution of Laplace equations, are considered. An indirect BEM formulation is applied to obtain surface source densities on domain boundaries of piecewise homogeneous, linear, isotropic materials. In the case of electrostatic field problems, the scalar electric potential $u(\boldsymbol{r})$ at an arbitrary point $\boldsymbol{r}$ is directly obtained from the surface charge density $\sigma$ on the surface $A$ of conductors or dielectrics by an evaluation of

$$u(\boldsymbol{r}) = \frac{1}{\epsilon_0}\int_A \sigma(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}A'. \tag{1}$$

$G$ is Green's function of Laplace equation

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{2}$$

A system of linear equations is obtained by an application of second order quadrilateral boundary elements and the Galerkin method [10]. The dense matrix is compressed using the fast multipole method (FMM) [7].

Here, isovalues of $u(\boldsymbol{r})$ are searched for a visualization with isosurfaces. The values of $u(\boldsymbol{r})$ can be directly obtained from (1). The evaluation of (1) is significantly accelerated, if the FMM is applied for the post-processing, too [8]. Then, an octree scheme is used to group all boundary elements. The integral in (1) is only evaluated for a small number of boundary elements in the near-field of an evaluation point $\boldsymbol{r}_{ep}$. However, singular and nearly singular integrals must be computed. The far-field of $\boldsymbol{r}_{ep}$ is taken into account by an evaluation of the local expansion in the octree cube of the evaluation point $C_{ep}$

$$u_f(\boldsymbol{r}_{ep}) = \frac{1}{4\pi}\sum_{n=0}^{L}\sum_{m=-n}^{n} L_n^m r^n Y_n^m(\theta,\varphi). \tag{3}$$

$L_n^m$ are the local coefficients in $C_{ep}$, $L$ is the order of series expansions, and $r$, $\theta$, $\varphi$ are spherical coordinates of $\boldsymbol{r}_{ep}$ with origin in the center of $C_{ep}$. $Y_n^m$ are spherical harmonics. In practice, a good accuracy is obtained for $L = 9$. That means, $u(\boldsymbol{r})$ is approximated in $C_{ep}$ with higher order polynomials and only a few coefficients. An approximation using linear volume elements would require a relatively large number of elements to achieve a comparable accuracy of $u(\boldsymbol{r})$.

### B. Meshfree isovalue search method

An isovalue search method requires to provide values of the examined field in total space or in practice in a large but limited spatial range. An important property of BEM is that due to (2) infinite space is taken into account exactly. The following method takes both aspects into account.

The total space is subdivided into two regions. One region, the inner domain $\Omega_i$, contains the boundary elements and requires carefully evaluations of compressed BEM integrals and the other region, the outer domain $\Omega_o$, is the surrounding free space, where the distance of all evaluation points to the boundary elements is large enough to only apply series expansions for the compressed evaluation of (1).

The first step is to create two dense octrees, one for $\Omega_i$ and one for $\Omega_o$. The root cube $C_{ri}$ of the inner octree $T_i$ is constructed with the help of the bounding box $B_e$ of all boundary elements. The center of $C_{ri}$ corresponds with the center of $B_e$ and its edge length is twice the largest edge length of $B_e$. Then, outside of $C_{ri}$ all prerequisites for an application of the multipole expansion

$$u_f(\boldsymbol{r}_{ep}) = \frac{1}{4\pi}\sum_{n=0}^{L}\sum_{m=-n}^{n} M_n^m \frac{Y_n^m(\theta,\varphi)}{r^{n+1}} \quad (4)$$

are fulfilled. The origin of the spherical coordinates is in the center of $C_{ri}$. The multipole coefficients $M_n^m$ contain information of sources on all boundary elements.

The cube $C_{ri}$ is subdivided into eight child cubes according the octree creation rules of the FMM. The difference to the classical FMM is that here empty cubes are created and stored, too. Empty cubes are only subdivided if boundary elements are assigned to their neighbor cubes and if these cubes are at a finer octree level than the empty cube.

The outer octree $T_o$ is constructed in dependency of $T_i$. The center of the root cube $C_{ro}$ of $T_o$ coincides with the center of $C_{ri}$. The edge length of $C_{ro}$ is

$$l_{ro} = n l_{ri}, \quad (5)$$

where $l_{ri}$ is the edge length of $C_{ri}$ and $n$ is a positive integer. The outer octree $T_o$ is subdivided into empty child cubes until the level with edge length of the child cubes of $C_{ri}$ is reached. The cubes, which would coincide with cubes of $T_i$ are omitted. Then, all cubes of $T_o$ surround $T_i$ and the field in all cubes of $T_o$ can be computed using only one multipole expansion (4). Of course, if very large surrounding air domains are considered, the order $L$ in (4) can be reduced for far distant cubes.

After the creation of the two octrees, the multipole expansions (4) and local expansions (3) of the FMM are computed in all cubes analogous to the classical FMM algorithm.

For an efficient octree based isovalue search, the value range of all cubes is determined first. Then, an efficient filtering of relevant cubes using the hierarchical tree structure is possible. Since extrema of $u(\boldsymbol{r})$ are lying on boundary elements, the values of $u(\boldsymbol{r})$ in all nodes of the boundary elements are determined first. Then the total range of $u(\boldsymbol{r})$ is known. The range of an octree cube is obtained from the values of $u(\boldsymbol{r})$ on boundary elements inside the cube and from the values of $u(\boldsymbol{r})$ in the corners of the cube. Note, it is necessary to take boundary element into account that are assigned to the cube and boundary elements of neighbor cubes that extend into the cube.

If no boundary elements are lying in a cube, the local expansion (3) suffices to compute $u(\boldsymbol{r})$. A significant difference to the well-known isovalue search method is that $u(\boldsymbol{r})$ is not linear interpolated between the corners of the cube but a higher order Taylor series expansion (3) is used. Hence, an optimization method, for instance a gradient based search method, is applied to determine the intersection points of the isosurface with the octree cube edges. An advantage of the presented approach is that the intersection points are determined with high accuracy. Furthermore, additional intersection points inside the cube can be computed with a refinement of the cube and isosurface with nearly arbitrary fine adjustable resolution are obtained.

If boundary elements are lying in the considered cube, the computation of intersection points is more expensive, because additional to the local expansion (3) nearly singular integrals in (1) must be evaluated. In that case, a mixture of the above described method and an interpolation using higher order volume elements is applied to achieve accurate results with reasonable computational costs.

## III. Numerical Example

A numerical example, which demonstrates the efficiency and accuracy of the presented isovalue search method, will be shown in the full paper.

## IV. References


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